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ASYMPTOTIC COMPARISON OF PARAMETERS ESTIMATES OF TWO-PARAMETER WEIBULL DISTRIBUTION

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Abstract

The Weibull distribution is frequently applied in various fields, ranging from economy, business, biology, to engineering. This paper aims at estimating the parameters of two-parameter Weibull distribution are determined. For this purpose, the method of quantiles (three different choices of quantiles) and Weibull probability plot method is utilized. The asymptotic covariance matrix of the parameter estimates is derived for both methods. For optimal random choices of quantiles, the variance, covariance and generalized variance is computed.
The main contribution of this study is the introduction of the best choice of percentiles for the method of quantiles and the joint asymptotic efficiency comparison of applied methods.

Keywords: Weibull distribution, estimation of parameters, method of quantiles, Weibull probability plot, asymptotic covariance matrix, joint asymptotic efficiency comparison

INTRODUCTION

Weibull distribution was firstly introduced by professor Waloddi Weibull in 1951, and it has found numerous applications in various fields, ranging from economy, business, biology to engineering.

For the estimation of parameters, the maximum likelihood method is frequently used. The estimates are computed iteratively, therefore the suitable choice of initial values is needed. Usually, the method of quantiles, the Weibull probability plot method, or the method of moments is used to determine the initial values.

In Newby (1980) the asymptotic covariance matrix of the method of moments was derived. Dubey (1967) introduced the estimation of parameters by the method of percentile.

In this paper, two methods for the estimation of parameters of two-parameter Weibull distribution are considered – the method of quantiles (three quantile choices) and Weibull probability plot method. For both methods, the asymptotic covariance matrix of the parameter estimates is derived. The variances, covariance and generalized variance (the determinant of asymptotic covariance matrix) are computed for the optimal and random choices of quantiles.

Two-parameter Weibull distribution

The two-parameter Weibull distribution is very often called standard Weibull distribution. Its cumulative distribution function has the form

$$F(x;\alpha,\beta)=\begin{cases}1-\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], & x\ge 0,\\ 0, & x<0,\end{cases} \tag{1}$$

$\alpha > 0$, $\beta > 0$. The parameter α is called the scale parameter, β is the shape parameter.

The probability density function is given by

$$f(x;\alpha,\beta)=\begin{cases}\frac{\beta x^{\beta-1}}{\alpha^{\beta}}\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], & x\ge 0,\\ 0, & x<0,\end{cases}$$

$\alpha > 0$, $\beta > 0$. The quantile function (sometimes called the percentile function) is given by

$$Q(p,\alpha,\beta)=\alpha\left[-\ln(1-p)\right]^{\frac{1}{\beta}}, \tag{2}$$

for p $\in$ [0;1] which we can compute as the inverse function of (1).

Asymptotic distribution of sample quantiles

In the next part, the asymptotic distribution of the empirical quantiles is needed. The primary sources of this subsection are papers by Mosteller (1946) and Dubey (1967) together with monography by Van der Vaart (1998).

Let X_1, X_2,..., X_n be a random sample of size from the two-parameter Weibull distribution. Let be the sample quantile corresponding to the given probability p.

The asymptotic distribution of the estimates can be used for comparison of different methods of estimation. Theorem 1 from Mosteller (1946) presents an approach to calculate the sample quantiles for the two-parameter Weibull distribution.

Theorem 1: Let the probability density function *f(x, α, β)* satisfy that it is continuous and does not vanish in the neighborhood of $x_{p_i,n}$ for $i = 1$,..., m ($1 \le m \le n$). Then asymptoticaly

$$\left(X_{p_1,n},\dots,X_{p_m,n}\right)\overset{as}{\sim} N_m\left(\left(Q(p_1,\alpha,\beta),\dots,Q(p_m,\alpha,\beta)\right),\boldsymbol{\Sigma}\right),$$

where $\boldsymbol{\Sigma}$ is symmetric variance matrix with diagonal elements

$$\sigma_i^2=\frac{p_i(1-p_i)}{nf^2\left(Q(p_i,\alpha,\beta)\right)}$$

for $i = 1$,..., m and off-diagonal elements

$$\sigma_{ij}=\frac{p_i(1-p_j)}{nf\left(Q(p_i,\alpha,\beta)\right)f\left(Q(p_j,\alpha,\beta)\right)}$$

for $1 \le i < j \le m$.

In the following sections, the distribution of the function of the sample quantiles is needed, and therefore Theorem 2 from Van der Vaart (1998) is introduced.

Theorem 2: Suppose that asymptotically

$$\sqrt{n}(\mathbf{Y}-\boldsymbol{\mu})\overset{as}{\sim} N_k(\mathbf{0},\boldsymbol{\Sigma}),$$

where $\mathbf{Y} = (Y_1,\dots, Y_k)$, $\mathbf{0} = (0,\dots,0)$ and suppose that $h_1,\dots,h_m$ are m real-valued functions of $\boldsymbol{\mu} = (\mu_1,\dots,\mu_k)$, defined and continuously differentiable in a neighbourhood ω of $\boldsymbol{\mu}$.

Denote a matrix $\mathbf{B}=\left\{\frac{\partial h_i}{\partial \mu_j}\right\}_{i=1,\dots,m,\, j=1,\dots,k}$ of partial derivatives. Then asymptotically

$$\left(\sqrt{n}[h_1(\mathbf{Y})-h_1(\boldsymbol{\mu})],\dots,\sqrt{n}[h_m(\mathbf{Y})-h_m(\boldsymbol{\mu})]\right)\overset{as}{\sim} N_m(\mathbf{0},\mathbf{B\Sigma B}^T). \tag{3}$$

MATERIALS AND METHODS – PARAMETER ESTIMATION

In this chapter, various ways of parameter estimation are established and their asymptotic distribution is derived.

Method of quantiles

In this section, the results of Murthy (2004) are utilized. The quantile function (2) is used.

The method of quantiles is based on the comparison of theoretical and empirical quantiles. Let $x_{p_i,n}$ be the empirical quantile corresponding to the given probability p_i, $i = 1,2$. The estimates of the parameters and are the solution of the system of equations

$$\begin{cases}X_{p_1,n}=\alpha\left[-\ln(1-p_1)\right]^{1/\beta}\\ X_{p_2,n}=\alpha\left[-\ln(1-p_2)\right]^{1/\beta}.\end{cases} \tag{4}$$

For the two-parametric Weibull model, several choices of quantiles for p_1 and p_2 are available. Some of them are considered in the following subsections.

The paper by Dubey (1967) focuses on the general case of the method of quantiles – namely the assumption

$$0 < p_1 < p_2 < 1.$$

The equation for parameter β is derived from the system of equations (4). Therefore, the estimation of β has the form

$$\hat{\beta} = \frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{\ln X_{p_1,n} - \ln X_{p_2,n}} \tag{5}$$

The estimation of parameter α is derived from the system of equations (4)

$$\hat{\alpha} = \exp\left[\frac{1}{2}\left(\ln X_{p_1,n} + \ln X_{p_2,n} - \frac{1}{\hat{\beta}}\ln\left[-\ln\left(1-p_1\right)\right] - \right.\right. \tag{6}$$

$$\left.\left. - \frac{1}{\hat{\beta}}\ln\left[-\ln\left(1-p_2\right)\right]\right)\right] =$$

$$= \prod_{i=1}^{2}\left(\frac{X_{p_i,n}}{\left[-\ln\left(1-p_i\right)\right]^{\frac{1}{\hat{\beta}}}}\right)^{\frac{1}{2}} \tag{7}$$

For the estimation of parameter α by (7), the estimation $\hat{\beta}$ must be used instead of the parameter β.

By substituting (5) into (6) the following equation is obtained

$$\hat{\alpha} = \exp\left[w \ln X_{p_1,n} + (1-w)\ln X_{p_2,n}\right],$$

where

$$w = 1 - \frac{\ln k_1}{k}, \quad k = \ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right],$$

$$k_i = -\ln\left(1-p_i\right) \tag{8}$$

for i = 1,2.

To derive the asymptotic covariance matrix of the random vector $(\hat{\alpha}, \hat{\beta})$ by Theorem 2, the first derivatives of functions h_1 and h_2 are needed. The functions have the form

$$h_1\left(Q_1, Q_2\right) = \exp\left[w \ln Q_1 + (1-w)\ln Q_2\right]$$

$$h_2\left(Q_1, Q_2\right) = \frac{\ln\left[-\ln\left(1-p_1\right)\right] - \ln\left[-\ln\left(1-p_2\right)\right]}{\ln Q_1 - \ln Q_2}$$

where Q_i is the theoretical quantile for i = 1,2 and w, k and k_i for i = 1,2 are specified in (8).

The asymptotic variances σ_1^2 and σ_1^2 of the sample quantiles $X_{p_1,n}$ and $X_{p_2,n}$ and their asymptotic covariance σ_{12} were computed from Theorem 1 with m = 2.

$$\sigma_1^2 = \frac{\alpha^2 p_1\left[-\ln\left(1-p_1\right)\right]^{2/\beta}}{n\beta^2\left(1-p_1\right)\left[-\ln\left(1-p_1\right)\right]^2} = \frac{\alpha^2 q_1 k_1^{2/\beta}}{n\beta^2 k_1^2}$$

$$\sigma_2^2 = \frac{\alpha^2 p_2\left[-\ln\left(1-p_2\right)\right]^{2/\beta}}{n\beta^2\left(1-p_2\right)\left[-\ln\left(1-p_2\right)\right]^2} = \frac{\alpha^2 q_2 k_2^{2/\beta}}{n\beta^2 k_2^2}$$

$$\sigma_{12} = \frac{\alpha^2 p_1\left[-\ln\left(1-p_1\right)\right]^{1/\beta}\left[-\ln\left(1-p_2\right)\right]^{1/\beta}}{n\beta^2\left(1-p_1\right)\left[-\ln\left(1-p_1\right)\right]\left[-\ln\left(1-p_2\right)\right]} = \frac{\alpha^2 q_1 k_1^{1/\beta} k_2^{1/\beta}}{n\beta^2 k_1 k_2}.$$

The asymptotic covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$ was computed from expression (3).

$$\begin{pmatrix} \mathrm{Var}\,\hat{\alpha} & \mathrm{Cov}\left(\hat{\alpha},\hat{\beta}\right) \\ \mathrm{Cov}\left(\hat{\alpha},\hat{\beta}\right) & \mathrm{Var}\,\hat{\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial Q_1} & \frac{\partial h_1}{\partial Q_2} \\ \frac{\partial h_2}{\partial Q_1} & \frac{\partial h_2}{\partial Q_2} \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial h_1}{\partial Q_1} & \frac{\partial h_2}{\partial Q_1} \\ \frac{\partial h_1}{\partial Q_2} & \frac{\partial h_2}{\partial Q_2} \end{pmatrix} \tag{9}$$

The functions h_1 and h_2 are defined, continuously differentiable for $\alpha > 0$, $\beta > 0$, and $0 < p_1 < p_2 < 1$, so the conditions of Theorem 2 are fulfilled. The derivatives can be seen on pages 19 and 20 in Konečná (2017).

It follows that asymptotically

$$\left(\hat{\alpha},\hat{\beta}\right) \overset{as}{\sim} N_2\left(\left(\alpha,\beta\right), \begin{pmatrix} \frac{\alpha^2}{n\beta^2}\, g\left(p_1,p_2\right) & \frac{\alpha}{nk^2}\, 1\left(p_1,p_2\right) \\ \frac{\alpha}{nk^2}\, 1\left(p_1,p_2\right) & \frac{\beta^2}{n}\, \phi\left(p_1,p_2\right) \end{pmatrix}\right),$$

where

$$g\left(p_1,p_2\right) = \frac{wq_1}{k_1}\left(\frac{w}{k_1} + 2\frac{1-w}{k_2}\right) + \frac{\left(1-w\right)^2 q_2}{k_2^2},$$

$$1\left(p_1,p_2\right) = \frac{q_1}{k_1}\left(\frac{k-\ln k_1}{k_2} - \frac{\ln k_1}{k_2} - \frac{k-\ln k_1}{k_1}\right) + \frac{q_2 \ln k_1}{k_2^2},$$

$$\phi\left(p_1,p_2\right) = \frac{q_1}{k_1^2} + \frac{q_2}{k_2^2} - 2\frac{q_1}{k_1 k_2}, \quad q_i = \frac{p_i}{\left(1-p_i\right)} \tag{10}$$

w, k and k_i for i = 1,2 are specified in (8).

Quantiles for Sum of Probabilities Equal to 1 – Special case 1

In the article by Pekasiewicz (2014) a method for choice of quantiles p_1 and p_2, such that $p_1 + p_2 = 1$ is developed.

The estimation of parameter β is derived from (5) after substitution $p_1 = p$ and $p_2 = 1 - p$.

$$\hat{\beta} = \frac{\ln\left[-\ln\left(1-p\right)\right] - \ln\left[-\ln p\right]}{\ln X_{p,n} - \ln X_{1-p,n}}$$

The estimation of parameter α is derived from (6) after substitution $p_1 = p$ and $p_2 = 1 - p$.

$$\hat{\alpha} = \exp\left[\frac{1}{2}\left(\ln X_{p,n} + \ln X_{1-p,n} - \frac{1}{\hat{\beta}}\ln\left[-\ln\left(1-p\right)\right] - \right.\right.$$
$$\left.\left. - \frac{1}{\hat{\beta}}\ln\left[-\ln p\right]\right)\right]$$

Special Quantiles – Special case 2

The estimation of the parameter α can be derived from the quantile function (2). The estimation is independent of the parameter β for quantile p_2 selected as

$$-\ln\left(1-p_2\right) = 1, \qquad (11)$$

so that $\hat{\alpha} = X_{p_2,n}$.

This implies that $p_2 = \left(1 - e^{-1}\right) = 0.632$. It means that the 0,632th quantile corresponds to the estimation of α.

$$\hat{\alpha} = X_{\left(1-e^{-1}\right),n}.$$

The estimation of parameter β is derived from (5) after substitution $p_2 = (1 - e^{-1})$

$$\hat{\beta} = \frac{\ln\left[-\ln\left(1-p_1\right)\right]}{\ln X_{p_1,n} - \ln X_{\left(1-e^{-1}\right),n}}$$

for $0 < p_1 < \left(1 - e^{-1}\right)$.

To derive the asymptotic covariance matrix of the random vector $(\hat{\alpha}, \hat{\beta})$ by Theorem 2, the first derivatives of functions h_1 and h_2 are computed. The functions have the form

$$h_1\left(Q_1, Q_2\right) = Q_2$$

$$h_2\left(Q_1, Q_2\right) = \frac{\ln\left[-\ln\left(1-p_1\right)\right]}{\ln Q_1 - \ln Q_2}$$

where Q_i is the theoretical quantile for $i = 1,2$.

Expressions for asymptotic variances and covariance are obtained from the equation (9) for the estimation of parameters of special quantiles. By (11) it is clear that $k_2 = 1$and and $k = \ln k_1$ thus asymptoticaly

$$\left(\hat{\alpha}, \hat{\beta}\right) \overset{as}{\sim} N_2\left(\left(\alpha, \beta\right), \begin{pmatrix} \frac{\alpha^2 q_2}{n\beta^2} & \frac{\alpha}{n\ln k_1}\left[-\frac{q_1}{k_1} + q_2\right] \\ \frac{\alpha}{n\ln k_1}\left[-\frac{q_1}{k_1} + q_2\right] & \frac{\beta^2}{n}\,\phi\left(p_1, p_2\right) \end{pmatrix}\right),$$

where ϕ, k_1 and q_i for $i = 1,2$ are defined in (8) and (10).

The functions h_1 and h_2 are defined, continuously differentiable for $\alpha > 0$, $\beta > 0$, and $0 < p_1 < p_2 < 1$, so the conditions of Theorem 2 are fulfilled. The derivatives can be found on page 22 in Konečná (2017).

Weibull Probability Plot Method

The graphical method of estimation of parameters for the random sample of size is based on the Weibull probability plot described in Dubey (1967).

The linear relation between $\ln Q\,(p,\alpha,\beta)$ and $\ln[-\ln(1-p)]$ is obtained by taking the logarithm of both sides of the quantile function (2)

$$\beta \ln Q\left(p, \alpha, \beta\right) - \beta \ln \alpha = \ln\left[-\ln\left(1-p\right)\right], \qquad (12)$$

which allows us to estimate the parameters α and β by least squares method often used in a linear regression.

For the Weibull probability plot the observed variable is z and the response variable is y.

$$z = \ln Q\left(p, \alpha, \beta\right)$$

$$y = \ln\left[-\ln\left(1-p\right)\right].$$

In the Weibull probability plot method, the empirical quantile of our data for the regular spacing of probability $p \in (0,1)$ is used, which can be set by the user with condition $p_i \neq p_j$ for $i \neq j$. For example, the values $p_1, p_2, \ldots, p_n$ are placed regularly on the interval ($p_i = 0.01 + (i - 1)(0.99 - 0.01)/n$). Comparing the form (12) with a linear dependence of against with intersection a and b slope the parameters α and β can be expressed as

$$\alpha = \exp\left[-\frac{a}{b}\right], \quad \beta = b. \qquad (13)$$

The following formulas are obtained by the least squares method

$$\hat{b} = \hat{\beta} = \frac{n\sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \ln k_i - \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \sum_{i=1}^{n} \ln k_i}{n\sum_{i=1}^{n} \left[\ln\left(X_{p_i,n}\right)\right]^2 - \left[\sum_{i=1}^{n} \ln\left(X_{p_i,n}\right)\right]^2},$$

$$\hat{a} = \frac{\sum_{i=1}^{n} \left[\ln\left(X_{p_i,n}\right)\right]^2 \sum_{i=1}^{n} \ln k_i - \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \ln k_i}{n \sum_{i=1}^{n} \left[\ln\left(X_{p_i,n}\right)\right]^2 - \left[\sum_{i=1}^{n} \ln\left(X_{p_i,n}\right)\right]^2},$$

where k_i for i = 1,2, ..., n are specified in (8), and n is the sample size, which must be bigger than 3. (This condition is obtained from the simple linear regression.)

Alternatively, the estimation of the parameter α is computed as

$$\hat{a} = \frac{1}{n} \sum_{i=1}^{n} \ln[-\ln(1 - p_i)] - \hat{b} \frac{1}{n} \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right)$$

and the form of $\hat{\alpha}$ and the first equation from (13) are used to estimate the parameter α as

$$\hat{\alpha} = \exp\left[-\frac{\sum_{i=1}^{n} \left[\ln\left(X_{p_i,n}\right)\right]^2 \sum_{i=1}^{n} \ln k_i - \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \ln k_i}{n \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \ln k_i - \sum_{i=1}^{n} \ln\left(X_{p_i,n}\right) \sum_{i=1}^{n} \ln k_i}\right].$$

Unlike (7), the previous estimate of the parameter α is not computed from just 2 quantiles.

To derive the asymptotic covariance matrix of the random vector $(\hat{\alpha}, \hat{\beta})$ by Theorem 2, the first derivatives of functions h_1 and h_2 are needed to compute. The functions have the form

$$h_1(Q_1, Q_2, \dots, Q_\mathrm{n}) = \exp\left[-\frac{\sum_{i=1}^{n} (\ln Q_i)^2 \sum_{i=1}^{n} \ln k_i - \sum_{i=1}^{n} \ln Q_i \sum_{i=1}^{n} \ln Q_i \ln k_i}{n \sum_{i=1}^{n} \ln Q_i \ln k_i - \sum_{i=1}^{n} \ln Q_i \sum_{i=1}^{n} \ln k_i}\right],$$

$$h_2(Q_1, Q_2, \dots, Q_\mathrm{n}) = \frac{n \sum_{i=1}^{n} \ln Q_i \ln k_i - \sum_{i=1}^{n} \ln Q_i \sum_{i=1}^{n} \ln k_i}{n \sum_{i=1}^{n} (\ln Q_i)^2 - \left(\sum_{i=1}^{n} \ln Q_i\right)^2},$$

where Q_i is the theoretical quantile for i = 1,2, ..., n and k_i for i = 1,2, ..., n are specified in (8).

From Theorem 1 with $m = n$ the asymptotic variances σ_1^2 of the sample quantiles $X_{p_1,n}$ and $X_{p_2,n}$ and their asymptotic covariance σ_{ij} are computed as

$$\sigma_i^2 = \frac{\alpha^2 p_i [-\ln(1 - p_i)]^{2/\beta}}{n\beta^2 (1 - p_i)[-\ln(1 - p_i)]^2} = \frac{\alpha^2 q_i k_i^{2/\beta}}{n\beta^2 k_i^2}$$

$$\sigma_{ij} = \frac{\alpha^2 p_i [-\ln(1 - p_i)]^{1/\beta} \left[-\ln\left(1 - p_j\right)\right]^{1/\beta}}{n\beta^2 (1 - p_i)[-\ln(1 - p_i)]\left[-\ln\left(1 - p_j\right)\right]} = \frac{\alpha^2 q_i k_i^{1/\beta} k_j^{1/\beta}}{n\beta^2 k_i k_j},$$

for $1 \le i < j \le n$. Notice that $\sigma_i^2 = \sigma_{ii}$ can be written for $i = j$.

The asymptotic covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$ is computed from expression (3)

$$\begin{pmatrix} \mathrm{Var}\ \hat{\alpha} & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{Var}\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial Q_1} & \cdots & \frac{\partial h_1}{\partial Q_n} \\ \frac{\partial h_2}{\partial Q_1} & \cdots & \frac{\partial h_2}{\partial Q_n} \end{pmatrix} \cdot \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial h_1}{\partial Q_1} & \frac{\partial h_2}{\partial Q_1} \\ \vdots & \vdots \\ \frac{\partial h_1}{\partial Q_n} & \frac{\partial h_2}{\partial Q_n} \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h_1}{\partial Q_i} \frac{\partial h_1}{\partial Q_j} \sigma_{ij} & \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h_1}{\partial Q_i} \frac{\partial h_2}{\partial Q_j} \sigma_{ij} \\ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h_1}{\partial Q_i} \frac{\partial h_2}{\partial Q_j} \sigma_{ij} & \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h_2}{\partial Q_i} \frac{\partial h_2}{\partial Q_j} \sigma_{ij} \end{pmatrix}.$$

Similarly, to the method of quantiles, the functions h_1 and h_2 are defined and continuously differentiable for $\alpha > 0, \beta > 0, p \in (0,1)$ and $n \ge 3$ so Theorem 2 is valid. It follows that asymptotically

$$(\hat{\alpha}, \hat{\beta}) \underset{\sim}{as} N_2\left((\alpha, \beta), \begin{pmatrix} \mathrm{Var}\ \hat{\alpha} & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{Var}\ \hat{\beta} \end{pmatrix}\right),$$

where

$$\text{Var}\ \hat{\alpha} = \left(n\sum_{l=1}^{n}\ln^2 k_l - \left(\sum_{l=1}^{n}\ln k_l\right)^2\right)^{-2}\cdot\frac{\alpha^2}{n\beta^2}\Bigg[\left(\sum_{l=1}^{n}\ln k_l\right)^2\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i$$
$$-\sum_{l=1}^{n}\ln k_l\sum_{l=1}^{n}\ln^2 k_l\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i - \sum_{l=1}^{n}\ln k_l\sum_{l=1}^{n}\ln^2 k_l\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i}$$
$$+\left(\sum_{l=1}^{n}\ln^2 k_l\right)^2\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}\Bigg],$$

$$\text{Var}\ \hat{\beta} = \left(n\sum_{l=1}^{n}\ln^2 k_l - \left(\sum_{l=1}^{n}\ln k_l\right)^2\right)^{-2}\cdot\frac{\beta^2}{n}\Bigg[\left(\sum_{l=1}^{n}\ln k_l\right)^2\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}$$
$$-n\sum_{l=1}^{n}\ln k_l\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i - n\sum_{l=1}^{n}\ln k_l\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i}$$
$$+n^2\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i\Bigg],$$

$$\text{Cov}\left(\hat{\alpha},\hat{\beta}\right) = \left(n\sum_{l=1}^{n}\ln^2 k_l - \left(\sum_{l=1}^{n}\ln k_l\right)^2\right)^{-2}\cdot\frac{\alpha}{n}\Bigg[\sum_{l=1}^{n}\ln k_l\sum_{l=1}^{n}\ln^2 k_l\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}$$
$$-n\sum_{l=1}^{n}\ln^2 k_l\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i} - \left(\sum_{l=1}^{n}\ln k_l\right)^2\sum_{j=1}^{n}\frac{1}{k_j}\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i$$
$$+n\sum_{l=1}^{n}\ln k_l\sum_{j=1}^{n}\frac{1}{k_j}\ln k_j\sum_{i=1}^{n}\frac{q_i}{k_i}\ln k_i\Bigg],$$

and k_i and q_i for $i = 1,2, \ldots, n$ are from (8) and (10).

RESULTS

Now the particular methods can be compared by deriving the asymptotic distribution of the parameter estimations together with their asymptotic covariance matrix.

In Tab. I, there are the values of the asymptotic variances, the asymptotic covariance and the asymptotic generalized variance (the determinant of the asymptotic covariance matrix) of the parameter estimates for the optimal or special quantiles of the method of quantiles. The values of Var $\hat{\alpha}$ and Cov $(\hat{\alpha}, \hat{\beta})$ are dependent on the parameter β.

General Case

The optimal choice of p_1 and p_2 was computed by the minimization of the generalized variance (the determinant of the asymptotic covariance matrix). According to Dubey (1967), the optimal choice of the quantiles was obtained as

$$p_1 = 0.23875930,\ p_2 = 0.92656148 \qquad (14)$$

Our results of minimization of determinant of the asymptotic covariance matrix $\mathbf{\Sigma}$ from (3) are

$$p_1 = 0.2624487,\ p_2 = 0.9162927 \qquad (15)$$

which was computed in R by R function One Dimensional Optimization (optimize). The asymptotic variances, the covariance, and the determinant of the asymptotic covariance matrix of the random vector $(\hat{\alpha}, \hat{\beta})$ have a similar value for these quantiles (the difference is smaller then 10^{-3}). The difference between p_1, p_2 given by (14) and p_1, p_2, given by (15) was probably caused by rounding error in R. This can be observed in Tab. I with the variance and covariance of the estimates.

Quantiles for the Sum of Probabilities Equal to 1

The asymptotic variances and the covariance for this method are similar to the general case.

The optimal choice is $p_1 = 0.1362754$ and $p_2 = 0.8637246$ with regard to the generalized variance of percentiles. The result was computed by the minimization of the generalized variance in R (function optimize). From Tab. I, it can be seen, that the method of quantiles with the sum of probabilities equal 1 is worse than the general case of this method.

Special Quantiles

In the paper by Seki and Yokoyama (1993), the authors choose p_2 = 0.632 and p = 0.31 proposed for the estimation of $\hat{\beta}$, which comes from the condition

$$\ln[-\ln(1 - p)] = -1.$$

From the previous equation, it can be obtained

$$p = \left(1 - e^{-e^{-1}}\right).$$

The optimal choice of percentile p_1 for p_2 = 0.632 can be computed based on the derivation of the generalized variance, which is p_1 = 0.1342915.

In Tab. I, it can be seen that the optimal choice of p_1 is better than the choice p from the paper by Seki and Yokoyama (1993) but it is still worse than the optimal choice from the general case.

Weibull Probability Plot Method

In Tab. II, the values of the asymptotic variances, the asymptotic covariance and the determinant of the asymptotic covariance matrix for the Weibull probability plot can be seen for some particular choices of n. The values of Var $\hat{\alpha}$ and Cov($\hat{\alpha}$, $\hat{\beta}$) are dependent on the parameter β.

In Tab. II, it is possible to see that with increasing size n of the random sample, the value of the determinant of the asymptotic covariance matrix decreases.

DISCUSSION

In the previous sections, the asymptotic covariance matrix of parameter estimation obtained by different estimation methods based on a transformation of the empirical quantiles of the two-parameters Weibull distribution, was derived. The optimal general case for the quantiles with p_1 = 0.23875930, p_2 = 0.92656148 is deduced to be the best case for the method of quantiles with respect to the generalized variance.

In Tab. III, it is possible to see that for small n the method of quantiles for the general case with quantiles p_1 = 0.23875930, p_2 = 0.92656148 has a smaller asymptotic generalized variance and therefore is better than the Weibull probability plot method. For $n \geq 23$ the determinant is smaller for the Weibull probability plot method. On the other hand, the disadvantage of the Weibull probability plot method is that it might be sensitive to outliers.

For completion, the studied method of parameters estimation can be compared with the maximum likelihood estimation which is one of the estimation method with the smallest the general variance.

The generalized variance for the maximum likelihood estimation (MLE) is according to Dubey (1967) or Konečná (2017)

$$\frac{n^2}{\alpha^2}|\Sigma| = \frac{n^2}{\alpha^2}\frac{6}{\pi^2} = \frac{n^2}{\alpha^2}0.6079.$$

All costed method of estimation can be compared by the joint asymptotic efficiency, which has form

$$E = \frac{\text{generalized variance of MLE}}{\text{generalized variance of estimate by method}}.$$

The joint asymptotic efficiency E takes values from $\langle 0;1\rangle$ for all α, β, p_i, and n. The closer to 1 the value E is, the more efficient the estimator is, compared to MLE.

From Newby (1980), it is known that the asymptotic efficiency of the method of moments is dependent on the shape parameter β. The joint efficiency is above 69 % for all the values of β being between 2 and 10. The joint efficiency varies from 0 to 61 % for $\beta \in \langle 0.3;1\rangle$.

As can be concluded from Tab. I, the asymptotic efficiency of the estimation by the general case of the method of quantiles is equal to 40.79 % and is independent on α, β, and n.

For the estimation by Weibull probability plot method, the asymptotic efficiency is dependent on n. For example, for n = 23 the efficiency is equal to 41.63 %. The biggest efficiency of 63.95 % is for n =96.

A comparison of the three methods by values of the joint asymptotic efficiency can be seen in Fig. 1. Moreover, a comparison of the three methods by graphic illustration of simulated data can be seen in Fig. 2 and Fig. 3. In Fig. 2, there are QQ-plots for one random sample, each one for one estimation method - the method of quantiles, Weibull probability plot method, and fitdist. In the method of quantiles or Weibull probability plot method, the equations (5) and (7) or (13) otherwise are used. The function fitdist is from the package fitdistrplus and uses the method of quantile in our case. The function carries out the quantile matching numerically, by minimization of the sum of squared differences between observed and theoretical quantiles. The weighted quantiles for this method were used (14) in our case.

I: *Asymptotic variances, covariance and generalized variance of parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ by three kinds of the method of quantiles - general case, quantiles for sum equal to 1 and special quantiles.*

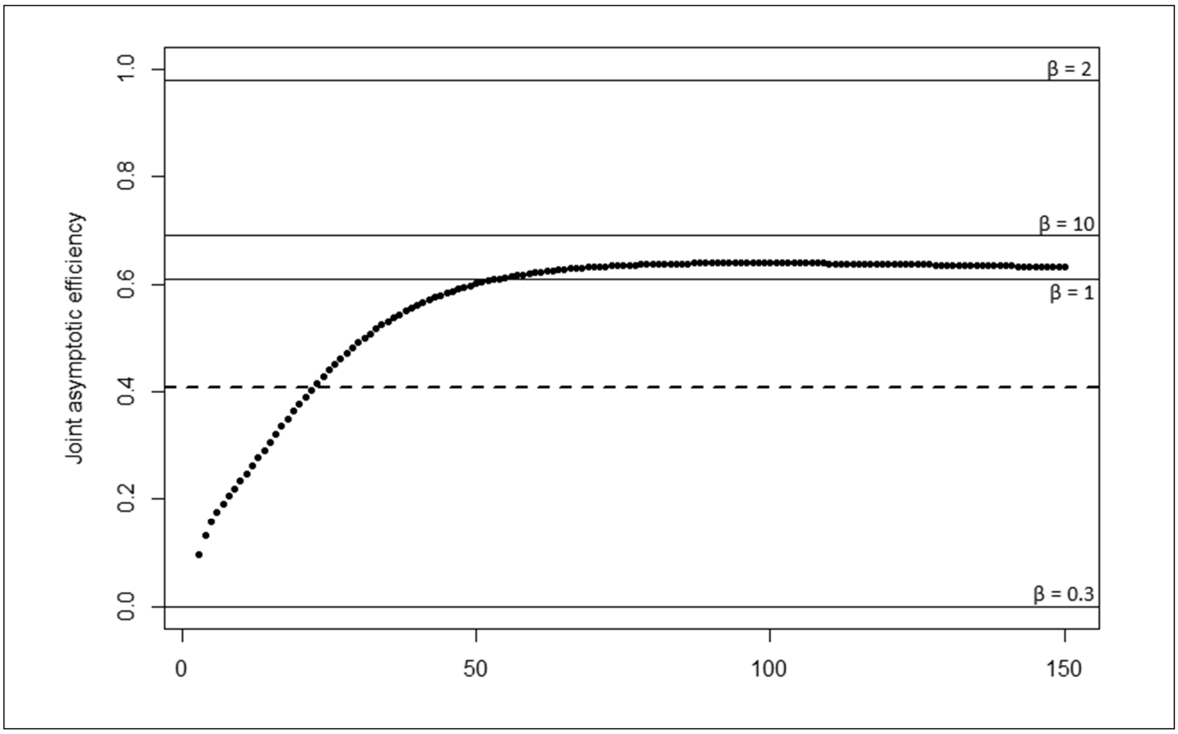
Method	Quantiles	$\frac{n\beta^2}{\alpha^2}$ Var $\hat{\alpha}$	$\frac{n}{\alpha}$ Cov$\left(\hat{\alpha},\hat{\beta}\right)$	$\frac{n}{\beta^2}$ Var $\hat{\beta}$	$\frac{n^2}{\alpha^2}\lvert\Sigma\rvert$
General case	p_1 = 0.23875930, p_2 = 0.92656148	1.016	1.588	0.351	1.4903
	p_1 = 0.2624487, p_2 = 0.9162927	1.0627	1.5233	0.3419	1.5019
Quantiles for sum equal to 1	p_1 = 0.1, p_2 = 0.9	1.1343	1.8162	0.5441	1.7639
	p_1 = 0.1362754, p_2 = 0.8637246	1.1544	1.587	0.3919	1.6784
	p_1 0.2, p_2 = 0.8	1.325	1.4358	0.2013	1.8619
	p_1 0.3, p_2 = 0.7	2.0152	1.4925	–0.1287	2.991
	p_1 0.4, p_2 = 0.6	4.3725	1.9284	–1.0023	7.4274
Special quantiles	p_1 = 0.1342915, p_2 = 0.632	1.8734	1.7174	–0.3314	3.1076
	p_1 = 0.0.31, p_2 = 0.632	2.6035	1.7174	–0.511	4.2101

II: *Asymptotic variances, covariance and generalized variance of parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ by Weibull probability plot method.*

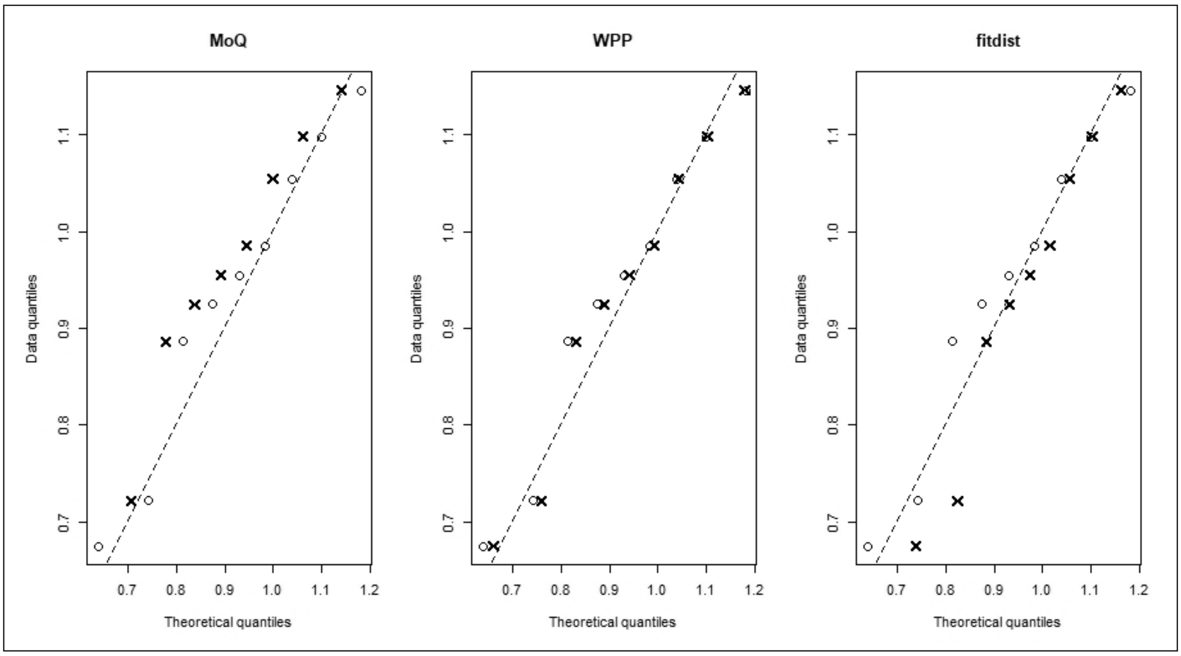
n	$\frac{1}{\beta^2}$ Var $\hat{\beta}$	$\frac{\beta^2}{\alpha^2}$ Var $\hat{\alpha}$	$\frac{1}{\alpha}$ Cov$\left(\hat{\alpha},\hat{\beta}\right)$	$\frac{1}{\alpha^2}\lvert\Sigma\rvert$
5	0.6211	0.2894	0.1612	0.1538
10	0.2296	0.1143	0.0137	0.0261
15	0.1154	0.0767	0.0051	0.0088
25	0.0483	0.0462	0.0045	0.0022
50	0.018	0.023	0.0033	0.0004

III: *Asymptotic variances, covariance and generalized variance of parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ by method of quantiles with general case (p_1 = 0.23875930, p_2 = 0.92656148) and Weibull probability plot method.*

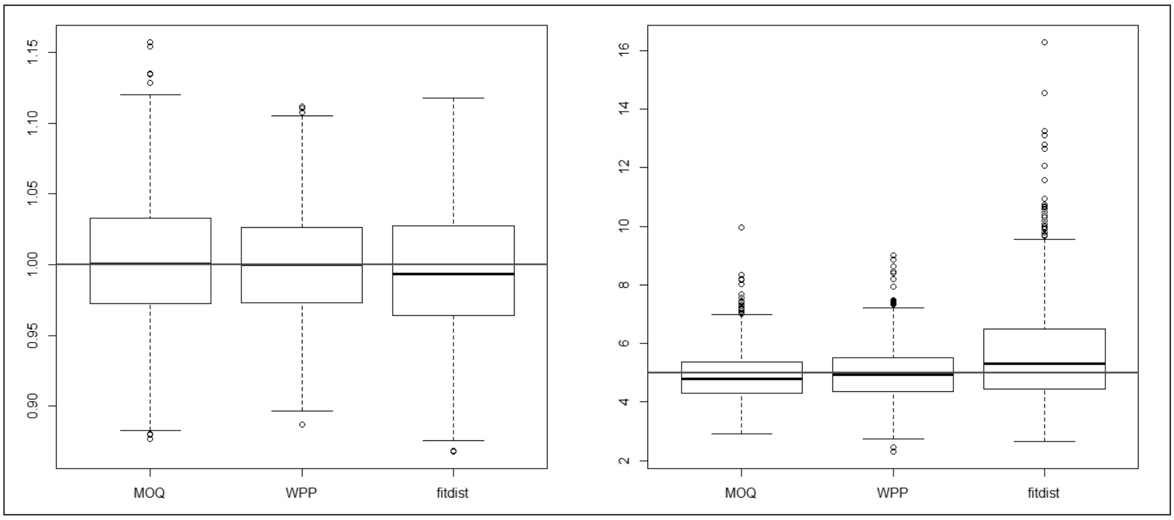
n	Method	$\frac{1}{\beta^2}$ Var $\hat{\beta}$	$\frac{\beta^2}{\alpha^2}$ Var $\hat{\alpha}$	$\frac{1}{\alpha}$ Cov$\left(\hat{\alpha},\hat{\beta}\right)$	$\frac{1}{\alpha^2}\lvert\Sigma\rvert$
5	**MoQ**	0.2032	0.3176	0.0702	0.0596
	WPP	0.6211	0.2894	0.1612	0.1538
10	**MoQ**	0.1016	0.1588	0.0351	0.0149
	WPP	0.2296	0.1143	0.0137	0.0261
15	**MoQ**	0.0677	0.1059	0.0234	0.0066
	WPP	0.1154	0.0767	0.0051	0.0088
23	**MoQ**	0.0442	0.069	0.0153	0.0028
	WPP	0.0554	0.0502	0.0046	0.0028
25	**MoQ**	0.0406	0.0635	0.014	0.0024
	WPP	0.0483	0.0462	0.0045	0.0022
50	**MoQ**	0.0203	0.0318	0.007	0.0006
	WPP	0.0181	0.023	0.0033	0.0004



1: *Graph of the joint asymptotic efficiency. The points are the values of the joint asymptotic efficiency of estimation by the Weibull probability plot method for sample of the size n. The dashed line is the value for the general case of the method of quantiles for* p_1 = 0.23875930 *and* p_2 = 0.92656148*. The continuous lines correspond to the value of the joint asymptotic efficiency for β the method of moments for equal to 0.3, 1, 2, and 10.*



2: *Q-Q plots. All QQ-plots of same random sample of size n = 30 from Weibull distribution with parameters α = 1, β = 5. The circles correspond to the true Weibull distribution with parameters α = 1 and β = 5 in each plot. In the first plot, the crosses represent the QQ-plot for Weibull distribution with parameters α = 0.9618 and β = 4.8496, which was estimated by the method of quantiles with the optimal choice of the quantiles (14). In the second plot, the crosses are the QQ-plot for Weibull distribution with parameters α = 1.0081 and β = 5.3073, which was estimated by the Weibull probability plot method. In the third plot, the crosses are the QQ-plot for Weibull distribution with parameters α = 1.0282 and β = 6.786 which was estimated by the function fitdist from the R package fitdistrplus and uses the method of quantile in this case.*



3: *Box plots of estimates of scale parameter (left) and shape parameter (right) based on 1000 generated random samples of size n = 30 from Weibull distribution with parameters α = 1, β = 5, by the method of quantiles with the optimal choice of the quantiles (14) (MoQ), the Weibull probability plot method (WPP), and the function fitdist from the R package fitdistrplus which used the method of quantiles.*

In Fig. 3, two Box plots are presented for each parameter estimation based on 1000 generated random samples of size 30 from Weibull distribution with $\alpha = 1$ and $\beta = 5$. In Tab. IV, the asymptotic generalized variance of estimates can be seen for each method. According to the results presented in this article, for this setting of parameters and size of the random sample, the best estimation method is the Weibull probability plot method, which can be seen in the graph presentation of the results of the simulation.

IV: *Sample mean, variances, covariance and generalized variances of parameter estimates based on 1000 generated random samples of size n = 30 from Weibull distribution with parameters α = 1, β = 5 by the method of quantiles with the optimal choice of the quantiles (14) (MoQ), the Weibull probability plot method (WPP), and the function fitdist from the R package fitdistrplus which use the method of quantiles.*

n = 30	MoQ		WPP		fitdist	
α = 1, β = 5	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
μ	1.0016	4.9121	1.0003	4.9779	0.9949	5.6141
Var	0.002	0.7476	0.0015	0.8485	0.002	2.8482
Cov	0.0093		0.0039		–0.0107	
Generalized variance	0.0014		0.0013		0.0057	

CONCLUSION

In this paper, the estimation of parameters by method of quantiles and the Weibull probability plot method for the two-parameter Weibull distribution are studied. For these estimations, the asymptotic variances, covariances, and the generalized variance were derived.

The results by Dubey (1967) were confirmed. Also, the chosen parameters $p_1 = 0.31$ and $p_2 = 0.632$ by Seki and Yokoyama (1993) have been demonstrated to be worse options than Dubey's results with respect to the generalized variance of the parameter estimates.

Obtained results can be used for computing the initial values for the maximum likelihood estimation method, which is an iteration method.

For a random sample of size $n \leq 22$, the best result is obtained by the method of quantiles for $p_1 = 0.23875930$ and $p_2 = 0.92656148$ from the article by Dubey (1967).

For a random sample of size $n \leq 23$, the best result is obtained by the Weibull probability plot method. It was shown in the paper, that the best choice between the method of moments from Newby (1980) and Weibull probability plot method with regard to the joint asymptotic efficiency depends on the sample size and the value of parameter β. Based on that, our suggestion is to estimate the parameters by the optimal general case of quantiles if $n \leq 22$, and by Weibull probability plot method otherwise. Then, it is possible to compare the joint asymptotic efficiency of the selected estimation and the estimates by the method of moments. Finally, if the joint asymptotic efficiency of the method of moment estimate is smaller, the initial estimate is recalculated by this method.

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